Measurable flows and perfect matchings in hyperfinite graphings

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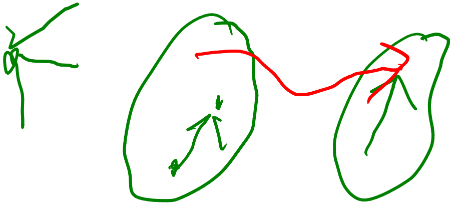
October 31, 2021

Graphings

Fix X a Polish space with borel probability measure μ and G a graph with $V(G) = X$.

- G is a *Borel* graph if $E(G)$ is a ~~Borel subset of X^2~~.
- G is *probability measure preserving* (pmp) if $\mu(A) = \mu(f(A))$ whenever $A \subseteq X$ is borel and f is a G-invariant borel injection.
- G is *locally finite* if $d(x) < \infty$ for every $x \in V(G)$.
- G is a *graphing* if it satisfies all of the above.



Hyperfinite graphings

G is (μ) *hyperfinite* if there is an increasing sequence G_n of induced borel subgraphs of G with finite connected components such that $\bigcup \cancel{V}(G_n) = \cancel{V}(G)$ (μ a.e.)

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- The Schreier graph of any free pmp action of an amenable group is μ hyperfinite.
- pmp hyperfinite trees have at most two ends a.e., so the Schreier graph of a free pmp action of F_d is never μ-hyperfinite.

Matchings and applications

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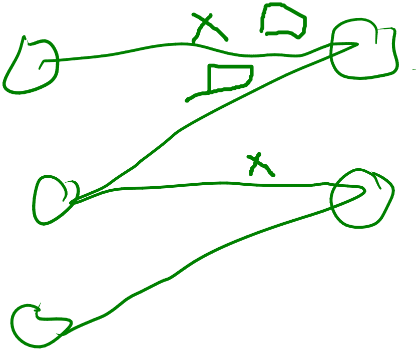

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Given an (usually pmp) action of Γ, we say A, B are Γ-equidecomposable if there is a partition $A = \bigcup_{i=1}^{n} A_i$ with $B = \bigcup_{i=1}^{n} \gamma_i A_i$ for some $\gamma_i \in \Gamma$

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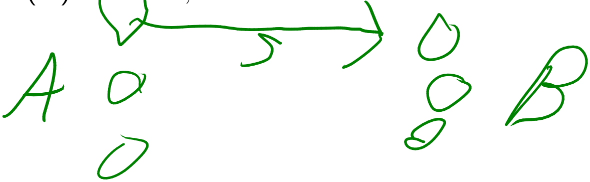
Given $A, B \subseteq X$ and $S \subseteq \Gamma$, $G = G(A, B, S)$ is the graph with $V(G) = A \cup B$, and aGb iff $as = b$ for some $s \in S$.



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A and B are Γ-equidecomposable iff $G(A, B, S)$ admits a perfect matching for some finite $S \subset \Gamma$.

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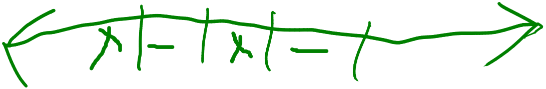
$\underline{T \times T}$ $\quad T^2$



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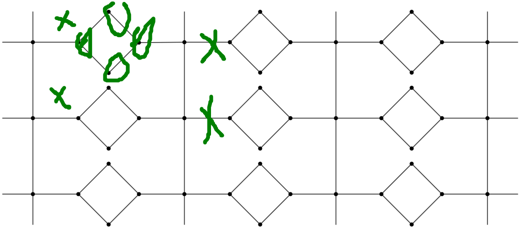
There are bipartite hyperfinite one-ended graphings with perfect matchings but no measurable perfect matching:



Measurable perfect matchings

Theorem (B., Kun, Sabok)

Suppose that G is a bipartite graph, and the set of G edges that appear in some perfect matching induces a one-ended hyperfinite graphing. Then G admits a measurable perfect matching a.e.

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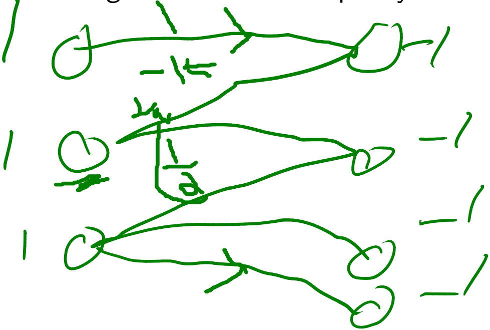
- Any d-regular, bipartite one-ended hyperfinite graphing admits a measurable perfect matching
- Any bipartite Schreier graph of a free pmp action of a one-ended amenable group admits a measurable perfect matching

Given $f : V(G) \to \mathbb{Z}$, an *f-flow* is a function $\phi : \vec{E}(G) \to \mathbb{R}$ if $\phi(x, y) = -\phi(y, x)$ and $f(x) = \sum_{y \in N(x)} \phi(x, y)$.

ϕ has capacity bounded by c if $\phi(x, y) \leq c(x, y)$.

Given a bipartite graph $G = (A, B, E)$ let $f(A) = 1, f(B) = -1$, and $c(a, b) \leq 1$ and $c(b, a) \leq 0$. Then G admits a perfect matching iff G admits an *integral* f-flow with capacity bounded by c.



warm up: measurable integral flows

Theorem (BKS)

Any one-ended hyperfinite graphing that admits an f-flow with capacity bounded by c admits a measurable integral f-flow with capacity bounded by $c + 3$.

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Any one-ended hyperfinite graphing that admits an f-flow with capacity bounded by c admits a measurable integral f-flow with capacity bounded by $c + 3$.

- This implies that if Γ is a one-ended amenable group and $A, B \subset X$ are "Γ-*uniform*" and Γ-equidecomposable, then they are equidecomposable using measurable pieces
- Cieśla and Sabok showed that hyperfinite graphings admit an f-flow with capacity c iff they admit a measurable f-flow with capacity c.

the proof part 1: tree tilings

Definition

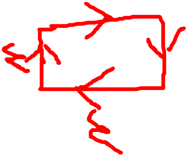
Given a graphing G, we say that a Borel collection $\mathcal{T}$ of finite connected subsets of $V(G)$ is a *one-ended tree tiling* if $\mathcal{T} \subseteq V(G)^{<\omega}$ satisfies

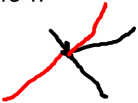
1. $\bigcup_{K \in \mathcal{T}} E(K) = E(G)$,
2. for every pair $K, L \in \mathcal{T}$ either $K \cap L = \emptyset$, or $K \cup N(K) \subseteq L$, or $L \cup N(L) \subseteq K$,
3. for every $K \in \mathcal{T}$ the induced subgraph on $K \setminus \bigcup_{K \supsetneq L \in \mathcal{T}} L$ is connected.

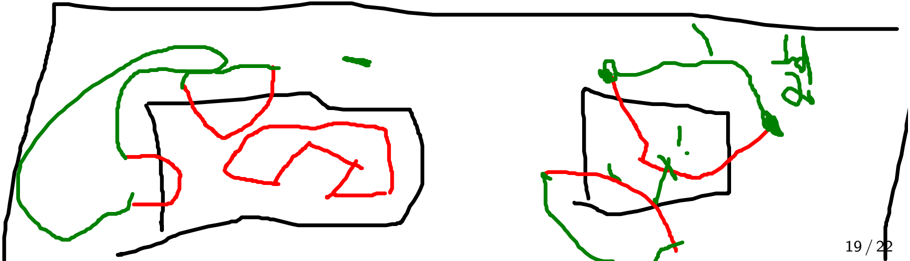
the proof part 2: more pictures

- Given an oreinted cycle, an ϵ-*circuit* sends ϵ flow along the cycle
- A rational is dyadic if its denominator is 2^n for some n

a)



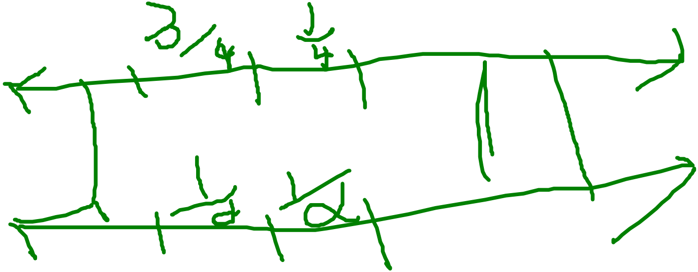




sketch of main theorem part 1

Let ϕ be a flow with a one-ended set of *free* edges

- Round along cycles until the graph is essentially acyclic
- Since G is hyperfinite, pmp, and no vertex can see only 1 non-integral edge we're left with bi-infinite lines to round
- if the flow value along these lines isn't $\frac{1}{2}/\frac{-1}{2}$ then we can select an end and round towards it



sketch of main theorem part 2

- Given flow ψ as in last slide, let $L(\psi) = \{e : \psi(e) \in \{\frac{1}{2}, \frac{-1}{2}\}$
- We want to find a ψ' with

$$\int_{\vec{e} \notin L(\psi)} |\psi'(\vec{e}) - \psi(\vec{e})| < \frac{1}{2} \int_{\vec{e} \in L(\psi)} |\psi'(\vec{e}) - \psi(\vec{e})|$$

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- Consider $(1 - \lambda)\psi + \lambda\phi$ for λ small
- Find borel family of cycles that cover half of the $e \in L(\psi)$ k times and edges not in $L(\psi)$ at most once (uses the one-ended tree tiling)
- adding randomly oriented ϵ-circuits and using Berry-Esseen gives the desired ψ'